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Freudenthal Duality in Gravity: from Groups of Type E_7 to Pre-Homogeneous Spaces

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ABSTRACT

Freudenthal duality can be defined as an anti-involutive, non-linear map acting on symplectic spaces. It was introduced in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields.

In this short review, I will consider its relation to the U-duality Lie groups of type E_7 in extended supergravity theories, and comment on the relation between the Hessian of the black hole entropy and the pseudo-Euclidean, rigid special (pseudo)Kähler metric of the pre-homogeneous spaces associated to the U-orbits.

Keywords : Extended Supergravity, Duality, Freudenthal Triple Systems, Special Kähler Geometry, Pre-Homogeneous Vector Spaces.

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1 Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions (*cfr. e.g.* [1]):

$$\mathcal{L}=-\frac{R}{2}+\frac{1}{2}g_{ij}\left(\varphi\right)\partial_{\mu}\varphi^{i}\partial^{\mu}\varphi^{j}+\frac{1}{4}I_{\Lambda\Sigma}\left(\varphi\right)F_{\mu\nu}^{\Lambda}F^{\Sigma|\mu\nu}+\frac{1}{8\sqrt{-G}}R_{\Lambda\Sigma}\left(\varphi\right)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{\Lambda}F_{\rho\sigma}^{\Sigma},\tag{1}$$

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that $\mathcal{L}$ may -but does not necessarily need to - be conceived as the bosonic sector of $D=4$ (*ungauged*) supergravity theory. Out of the Abelian two-form field strengths F^{Λ}'s, one can define their duals G_{Λ}, and construct a symplectic vector :

$$H:=\left(F^{\Lambda},G_{\Lambda}\right)^{T},\quad {}^{*}G_{\Lambda|\mu\nu}:=2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.\tag{2}$$

We then consider the simplest solution of the equations of motion deriving from $\mathcal{L}$, namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric [2]

$$ds^{2}=-e^{2U(\tau)}dt^{2}+e^{-2U(\tau)}\left[\frac{d\tau^{2}}{\tau^{4}}+\frac{1}{\tau^{2}}\left(d\theta^{2}+\sin\theta d\psi^{2}\right)\right],\tag{3}$$

where $\tau:=-1/r$. Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity S_{∞}^{2} in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector $\mathcal{Q}$:

$$p^{\Lambda}\ :=\frac{1}{4\pi}\int_{S_{\infty}^{2}}F^{\Lambda},\quad q_{\Lambda}:=\frac{1}{4\pi}\int_{S_{\infty}^{2}}G_{\Lambda},\tag{4}$$

$$\mathcal{Q}\ :=\left(p^{\Lambda},q_{\Lambda}\right)^{T}.\tag{5}$$

Then, by exploiting the symmetries of the background (3), the Lagrangian (1) can be dimensionally reduced from $D=4$ to $D=1$, obtaining a 1-dimensional effective Lagrangian ($'$ $:=d/d\tau$) [3]:

$$\mathcal{L}_{D=1}=\left(U'\right)^{2}+g_{ij}\left(\varphi\right)\varphi^{i\prime}\varphi^{j\prime}+e^{2U}V_{BH}\left(\varphi,\mathcal{Q}\right)\tag{6}$$

along with the Hamiltonian constraint [3]

$$\left(U'\right)^{2}+g_{ij}\left(\varphi\right)\varphi^{i\prime}\varphi^{j\prime}-e^{2U}V_{BH}\left(\varphi,\mathcal{Q}\right)=0.\tag{7}$$

The so-called "effective black hole potential" V_{BH} appearing in (6) and (7) is defined as [3]

$$V_{BH}\left(\varphi,\mathcal{Q}\right):=-\frac{1}{2}\mathcal{Q}^{T}\mathcal{M}\left(\varphi\right)\mathcal{Q},\tag{8}$$

in terms of the symplectic and symmetric matrix [1]

$$\mathcal{M}\ :=\left(\begin{array}{cc}\mathbb{I}&-R\\0&\mathbb{I}\end{array}\right)\left(\begin{array}{cc}I&0\\0&I^{-1}\end{array}\right)\left(\begin{array}{cc}\mathbb{I}&0\\-R&\mathbb{I}\end{array}\right)=\left(\begin{array}{cc}I+RI^{-1}R&-RI^{-1}\\-I^{-1}R&I^{-1}\end{array}\right),\tag{9}$$

$$\mathcal{M}^{T}=\mathcal{M};\quad\mathcal{M}\Omega\mathcal{M}=\Omega,\tag{10}$$

where $\mathbb{I}$ denotes the identity, and $R\left(\varphi\right)$ and $I\left(\varphi\right)$ are the scalar-dependent matrices occurring in (1); moreover, Ω stands for the symplectic metric ($\Omega^{2}=-\mathbb{I}$). Note that, regardless of the invertibility of $R\left(\varphi\right)$ and as a consequence of the physical consistence of the kinetic vector matrix $I\left(\varphi\right)$, $\mathcal{M}$ is negative-definite; thus, the effective black hole potential (8) is positive-definite.

By virtue of the matrix $\mathcal{M}$, one can introduce a (scalar-dependent) *anti-involution* $\mathcal{S}$ in any Maxwell-Einstein-scalar theory described by (1) with a symplectic structure Ω, as follows :

$$\begin{aligned} \mathcal{S}(\varphi) &:= \Omega\mathcal{M}(\varphi); && (11) \\ \mathcal{S}^2(\varphi) &= \Omega\mathcal{M}(\varphi)\Omega\mathcal{M}(\varphi) = \Omega^2 = -\mathbb{I}; && (12) \end{aligned}$$

in turn, this allows to define an anti-involution on the dyonic charge vector $\mathcal{Q}$, which has been called (scalar-dependent) *Freudenthal duality* [4, 5, 6]:

$$\begin{aligned} \mathfrak{F}(\mathcal{Q};\varphi) &:= -\mathcal{S}(\varphi)\mathcal{Q}; && (13) \\ \mathfrak{F}^2 &= -\mathbb{I}, \quad (\forall\{\varphi\}). && (14) \end{aligned}$$

By recalling (8) and (11), the action of $\mathfrak{F}$ on $\mathcal{Q}$, defining the so-called (φ-dependent) Freudenthal dual of $\mathcal{Q}$ itself, can be related to the symplectic gradient of the effective black hole potential V_{BH} :

$$\mathfrak{F}(\mathcal{Q};\varphi) = \Omega\frac{\partial V_{BH}(\varphi,\mathcal{Q})}{\partial\mathcal{Q}}. \tag{15}$$

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (3) (denoted below by the subscript "H"); indeed

$$\begin{aligned} \partial_\varphi V_{BH} &= 0 \Leftrightarrow \lim_{\tau\to-\infty}\varphi^i(\tau) = \varphi^i_H(\mathcal{Q}); && (16) \\ S_{BH}(\mathcal{Q}) &= \frac{A_H}{4} = \pi \left.V_{BH}\right|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H(\mathcal{Q})\mathcal{Q}, && (17) \end{aligned}$$

where S_{BH} and A_H respectively denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathcal{M}_H$ is defined as

$$\mathcal{M}_H(\mathcal{Q}) := \lim_{\tau\to-\infty}\mathcal{M}(\varphi(\tau)). \tag{18}$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathfrak{F}_H$ as the horizon limit of (13) :

$$\widetilde{\mathcal{Q}} \equiv \mathfrak{F}_H(\mathcal{Q}) := \lim_{\tau\to-\infty}\mathfrak{F}(\mathcal{Q};\varphi(\tau)) = -\Omega\mathcal{M}_H(\mathcal{Q})\mathcal{Q} = \frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}. \tag{19}$$

Remarkably, the (horizon) Freudenthal dual of $\mathcal{Q}$ is nothing but ($1/\pi$ times) the symplectic gradient of the Bekenstein-Hawking black hole entropy S_{BH}; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $\mathcal{Q}$. As a result, $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}(\mathcal{Q})$ is generally a complicated (non-linear) function, homogeneous of degree one in $\mathcal{Q}$.

It can be proved that the entropy S_{BH} itself is invariant along the flow in the charge space $\mathcal{Q}$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $\mathcal{Q}$ itself :

$$S_{BH}(\mathcal{Q}) = S_{BH}(\mathfrak{F}_H(\mathcal{Q})) = S_{BH}\left(\frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}\right) = S_{BH}\left(\widetilde{\mathcal{Q}}\right). \tag{20}$$

It is here worth pointing out that this invariance is pretty remarkable : the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9, 10, 11].

2 Groups of Type E_7

The concept of Lie groups *of type* E_7 as introduced in the 60s by Brown [12], and then later developed *e.g.* by [13, 14, 15, 16, 17].

Starting from a pair $(G, \mathbf{R})$ made of a Lie group G and its faithful representation $\mathbf{R}$, the three axioms defining $(G, \mathbf{R})$ as a group of type E_7 read as follows :

1. Existence of a (unique) symplectic invariant structure Ω in $\mathbf{R}$:
$$\exists ! \Omega \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}, \tag{21}$$
which then allows to define a symplectic product $\langle \cdot, \cdot \rangle$ among two vectors in the representation space $\mathbf{R}$ itself :
$$\langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = - \langle Q_2, Q_1 \rangle . \tag{22}$$

2. Existence of (unique) rank-4 completely symmetric invariant tensor (K-tensor) in $\mathbf{R}$:
$$\exists ! K \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s , \tag{23}$$
which then allows to define a degree-4 invariant polynomial I_4 in $\mathbf{R}$ itself :
$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \tag{24}$$

3. Defining a triple map T in $\mathbf{R}$ as
$$T \ : \ \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}; \tag{25}$$
$$\langle T (Q_1, Q_2, Q_3), Q_4 \rangle \ : = K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \tag{26}$$
it holds that
$$\langle T (Q_1, Q_1, Q_2), T (Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \tag{27}$$
This property makes a group of type E_7 amenable to a description as an automorphism group of a *Freudenthal triple system* (or, equivalently, as the conformal groups of the underlying Jordan triple system - whose a Jordan algebra is a particular case -).

All electric-magnetic duality (U-duality[1]) groups of $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type E_7. Among these, degenerate groups of type E_7 are those in which the K-tensor is actually reducible, and thus I_4 is the square of a quadratic invariant polynomial I_2. In fact, in general, in theories with electric-magnetic duality groups of type E_7 holds that
$$S_{BH} = \pi \sqrt{|I_4 (\mathcal{Q})|} = \pi \sqrt{|K_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q|}, \tag{28}$$
whereas in the case of degenerate groups of type E_7 it holds that $I_4 (\mathcal{Q}) = (I_2 (\mathcal{Q}))^2$, and therefore the latter formula simplifies to
$$S_{BH} = \pi \sqrt{|I_4 (\mathcal{Q})|} = \pi |I_2 (\mathcal{Q})| . \tag{29}$$

[1]Here U-duality is referred to as the "continuous" symmetries of [18]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [19].

J_3	G_4	$\mathbf{R}$	$\mathcal{N}$
$J_3^{\mathbb{O}}$	$E_{7(-25)}$	$\mathbf{56}$	2
$J_3^{\mathbb{O}_s}$	$E_{7(7)}$	$\mathbf{56}$	8
$J_3^{\mathbb{H}}$	$SO^*(12)$	$\mathbf{32}$	2, 6
$J_3^{\mathbb{H}_s}$	$SO(6,6)$	$\mathbf{32}$	0
$J_3^{\mathbb{C}}$	$SU(3,3)$	$\mathbf{20}$	2
$J_3^{\mathbb{C}_s}$	$SL(6,\mathbb{R})$	$\mathbf{20}$	0
$M_{1,2}(\mathbb{O})$	$SU(1,5)$	$\mathbf{20}$	5
$J_3^{\mathbb{R}}$	$Sp(6,\mathbb{R})$	$\mathbf{14}'$	2
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ (STU)	$[SL(2,\mathbb{R})]^3$	$(\mathbf{2},\mathbf{2},\mathbf{2})$	2
$\mathbb{R}$ (T^3)	$SL(2,\mathbb{R})$	$\mathbf{4}$	2

Table 1: Simple, non-degenerate groups G related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras J_3. In general, $G \cong Conf(J_3) \cong Aut(\mathfrak{M}(J_3))$ (see *e.g.* [20, 21, 22] for a recent introduction, and a list of Refs.). $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ are the corresponding split forms. Note that the G related to split forms $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ is the *maximally non-compact* (*split*) real form of the corresponding compact Lie group. $M_{1,2}(\mathbb{O})$ is the Jordan triple system generated by 2×1 vectors over $\mathbb{O}$ [23]. Note that the STU model, based on $J_3 = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, has a *semi-simple* G_4, but its *triality symmetry* [24] renders it "effectively simple". The $D = 5$ uplift of the T^3 model based on $J_3 = \mathbb{R}$ is the *pure* $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^{\mathbb{H}}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are *"twin"*, namely they share the very same bosonic sector [23, 25, 26, 27].

Simple, non-degenerate groups of type E_7 relevant to $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are reported in Table 1.

Semi-simple, non-degenerate groups of type E_7 of the same kind are given by $G = SL(2,\mathbb{R})\times$

$SO(2,n)$ and $G = SL(2,\mathbb{R}) \times SO(6,n)$, with $\mathbf{R} = (\mathbf{2}, \mathbf{2}+\mathbf{n})$ and $\mathbf{R} = (\mathbf{2}, \mathbf{6}+\mathbf{n})$, respectively relevant for $\mathcal{N}=2$ and $\mathcal{N}=4$ supergravity.

Moreover, degenerate (simple) groups of type E_7 relevant to the same class of theories are $G = U(1,n)$ and $G = U(3,n)$, with complex fundamental representations $\mathbf{R} = \mathbf{n}+\mathbf{1}$ and $\mathbf{R} = \mathbf{3}+\mathbf{n}$, respectively relevant for $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity [16].

The classification of groups of type E_7 is still an open problem, even if some progress have been recently made *e.g.* in [28] (in particular, *cfr.* Table D therein).

In all the aforementioned cases, the scalar manifold is a *symmetric* cosets $\frac{G}{H}$, where H is the maximal compact subgroup (with symmetric embedding) of G. Moreover, the K-tensor can generally be expressed as [17]

$$K_{MNPQ} = -\frac{n(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n\,(2n+1)}\Omega_{M(P}\Omega_{Q)N}\right], \tag{30}$$

where $\dim \mathbf{R} = 2n$ and $\dim G = d$, and t^{α}_{MN} denotes the symplectic representation of the generators of G itself. Thus, the horizon Freudenthal duality can be expressed in terms of the K-tensor as follows [4]:

$$\mathfrak{F}_H\left(\mathcal{Q}\right)_M \equiv \widetilde{\mathcal{Q}}_M = \frac{\partial\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}}{\partial \mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q, \tag{31}$$

where $\epsilon := I_4/\left|I_4\right|$; note that the horizon Freudenthal dual of a given symplectic dyonic charge vector $\mathcal{Q}$ is well defined only when $\mathcal{Q}$ is such that $I_4\left(\mathcal{Q}\right) \neq 0$. Consequently, the invariance (20) of the black hole entropy under the the horizon Freudenthal duality can be recast as the invariance of I_4 itself :

$$I_4\left(\mathcal{Q}\right) = I_4\left(\widetilde{\mathcal{Q}}\right) = I_4\left(\Omega\frac{\partial\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}}{\partial \mathcal{Q}}\right). \tag{32}$$

In absence of "flat directions" at the attractor points (namely, of unstabilized scalar fields at the horizon of the black hole), and for $I_4 > 0$, the expression of the matrix $\mathcal{M}_H\left(\mathcal{Q}\right)$ at the horizon can be computed to read

$$\mathcal{M}_{H|MN}(\mathcal{Q}) = -\frac{1}{\sqrt{I_4}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \mathcal{Q}_M\mathcal{Q}_N\right), \tag{33}$$

and it is invariant under horizon Freudenthal duality :

$$\mathfrak{F}_H\left(\mathcal{M}_H\right)_{MN} := \mathcal{M}_{H|MN}(\widetilde{\mathcal{Q}}) = \mathcal{M}_{H|MN}(\mathcal{Q}). \tag{34}$$

3 Duality Orbits, Rigid Special Kähler Geometry and Pre-Homogeneous Vector Spaces

For $I_4 > 0$, $\mathcal{M}_H\left(\mathcal{Q}\right)$ given by (33) is one of the two possible solutions to the set of equations [29]

$$\begin{cases} M^T\left(\mathcal{Q}\right)\Omega M\left(\mathcal{Q}\right) = \epsilon\Omega; \\ \\ M^T\left(\mathcal{Q}\right) = M\left(\mathcal{Q}\right); \\ \\ \mathcal{Q}^T M\left(\mathcal{Q}\right)\mathcal{Q} = -2\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}, \end{cases} \tag{35}$$

which describes symmetric, purely $\mathcal{Q}$-dependent structures at the horizon; they are symplectic or anti-symplectic, depending on whether $I_4 > 0$ or $I_4 < 0$, respectively. Since in the class

of (super)gravity $D=4$ theories discussed the sign of I_4 actually determines a stratification of the representation space $\mathbf{R}$ of charges into distinct orbits of the action of G into $\mathbf{R}$ itself (usually named duality orbits), the symplectic or anti-symplectic nature of the solutions to the system (35) is G-invariant, and supported by the various duality orbits of G (in particular, by the so-called "large" orbits, for which I_4 is non-vanishing).

One of the two possible solutions to the system (35) reads [29]

$$\begin{aligned} M_+(\mathcal{Q}) &= -\frac{1}{\sqrt{|I_4|}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \epsilon\mathcal{Q}_M\mathcal{Q}_N\right); \\ \mathfrak{F}_H\left(M_+\right)_{MN} &:= M_{+|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{+|MN}(\mathcal{Q}). \end{aligned}$$

For $\epsilon=+1 \Leftrightarrow I_4>0$, it thus follows that

$$M_+(\mathcal{Q}) = \mathcal{M}_H\left(\mathcal{Q}\right), \tag{36}$$

as anticipated.

On the other hand, the other solution to system (35) reads [29]

$$M_-\left(\mathcal{Q}\right) = \frac{1}{\sqrt{|I_4|}}\left(\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q\right); \tag{37}$$

$$\mathfrak{F}_H\left(M_-\right)_{MN} := M_{-|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{-|MN}(\mathcal{Q}). \tag{38}$$

By recalling the definition of I_4 (24), it is then immediate to realize that $M_-\left(\mathcal{Q}\right)$ is the (opposite of the) Hessian matrix of ($1/\pi$ times) the black hole entropy S_{BH} :

$$M_{-|MN}\left(\mathcal{Q}\right) = -\partial_M\partial_N\sqrt{|I_4|} = -\frac{1}{\pi}\partial_M\partial_N S_{BH}. \tag{39}$$

The matrix $M_-\left(\mathcal{Q}\right)$ is the (opposite of the) pseudo-Euclidean metric of a non-compact, non-Riemannian rigid special Kähler manifold related to the duality orbit of the black hole electromagnetic charges (to which $\mathcal{Q}$ belongs), which is an example of pre-homogeneous vector space (PVS) [30]. In turn, the nature of the rigid special manifold may be Kähler or pseudo-Kähler, depending on the existence of a $U(1)$ or $SO(1,1)$ connection[2].

In order to clarify this statement, let us make two examples within maximal $\mathcal{N}=8$, $D=4$ supergravity. In this theory, the electric-magnetic duality group is $G=E_{7(7)}$, and the representation in which the e.m. charges sit is its fundamental $\mathbf{R}=\mathbf{56}$. The scalar manifold has rank-7 and it is the real symmetric coset[3] $G/H=E_{7(7)}/SU(8)$, with dimension 70.

1. The unique duality orbit determined by the G-invariant constraint $I_4>0$ is the 55-dimensional non-symmetric coset

$$\mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}. \tag{40}$$

   By customarily assigning positive (negative) signature to non-compact (compact) generators, the pseudo-Euclidean signature of $\mathcal{O}_{I_4>0}$ is $(n_+,n_-)=(30,25)$. In this case, $M_-\left(\mathcal{Q}\right)$ given by (39) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special Kähler non-symmetric manifold

$$\mathbf{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}\times\mathbb{R}^+, \tag{41}$$

   with signature $(n_+,n_-)=(30,26)$, thus with character $\chi:=n_+-n_-=4$. Through a conification procedure (amounting to modding out[4] $\mathbb{C}\cong SO(2)\times SO(1,1)\cong U(1)\times\mathbb{R}^+$,

[2]For a thorough introduction to special Kähler geometry, see *e.g.* [31].

[3]To be more precise, it is worth mentioning that the actual relevant coset manifold is $E_{7(7)}/[SU(8)/\mathbb{Z}_2]$, because spinors transform according to the double cover of the stabilizer of the scalar manifold (see *e.g.* [32, 33], and Refs. therein).

[4]The signature along the $\mathbb{R}^+$-direction is negative [29].

G	V	n	isotropy alg.	degree	interpr. $D=4$
$SL(2,\mathbb{C})$	$S^3\mathbb{C}^2$	1	0	4	$\mathcal{N}=2,\mathbb{R}\ (T^3)$
$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})^{\oplus 2}$	4	$\mathcal{N}=2, J_3^{\mathbb{C}}$ $\mathcal{N}=0, J_3^{\mathbb{C}_s}$ $\mathcal{N}=5, M_{1,2}(\mathbb{O})$
$SL(7,\mathbb{C})$	$\Lambda^3\mathbb{C}^7$	1	$\mathfrak{g}_2^{\mathbb{C}}$	7	
$SL(8,\mathbb{C})$	$\Lambda^3\mathbb{C}^8$	1	$\mathfrak{sl}(3,\mathbb{C})$	16	
$SL(3,\mathbb{C})$	$S^2\mathbb{C}^3$	2	0	6	
$SL(5,\mathbb{C})$	$\Lambda^2\mathbb{C}^5$	3 4	$\mathfrak{sl}(2,\mathbb{C})$ 0	5 10	
$SL(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^6$	2	$\mathfrak{sl}(2,\mathbb{C})^{\oplus 3}$	6	
$SL(3,\mathbb{C})^{\otimes 2}$	$\mathbb{C}^3\otimes\mathbb{C}^3$	2	$\mathfrak{gl}(1,\mathbb{C})^{\oplus 2}$	6	
$Sp(6,\mathbb{C})$	$\Lambda_0^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})$	4	$\mathcal{N}=2, J_3^{\mathbb{R}}$
$Spin(7,\mathbb{C})$	$\mathbb{C}^8$	1 2 3	$\mathfrak{g}_2^{\mathbb{C}}$ $\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{so}(2,\mathbb{C})$ $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$	2 2 2	
$Spin(9,\mathbb{C})$	$\mathbb{C}^{16}$	1	$\mathfrak{spin}(7,\mathbb{C})$	2	
$Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	2 3	$\mathfrak{g}_2^{\mathbb{C}}\oplus\mathfrak{sl}(2,\mathbb{C})$ $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$	2 4	
$Spin(11,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(5,\mathbb{C})$	4	
$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(6,\mathbb{C})$	4	$\mathcal{N}=2,6, J_3^{\mathbb{H}}$ $\mathcal{N}=0, J_3^{\mathbb{H}_s}$
$Spin(14,\mathbb{C})$	$\mathbb{C}^{64}$	1	$\mathfrak{g}_2^{\mathbb{C}}\oplus\mathfrak{g}_2^{\mathbb{C}}$	8	
$G_2^{\mathbb{C}}$	$\mathbb{C}^7$	1 2	$\mathfrak{sl}(3,\mathbb{C})$ $\mathfrak{gl}(2,\mathbb{C})$	2 2	
$E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	1 2	$\mathfrak{f}_4^{\mathbb{C}}$ $\mathfrak{so}(8,\mathbb{C})$	3 6	
$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	1	$\mathfrak{e}_6^{\mathbb{C}}$	4	$\mathcal{N}=2, J_3^{\mathbb{O}}$ $\mathcal{N}=8, J_3^{\mathbb{O}_s}$

Table 2: Non-generic, nor irregular PVS with simple G, of type 2 (in the complex ground field). To avoid discussing the finite groups appearing, the list presents the Lie algebra of the isotropy group rather than the isotropy group itself [34]. The interpretation (of suitable real, non-compact slices) in $D=4$ theories of Einstein gravity is added; remaining cases will be investigated in a forthcoming publication

one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special Kähler symmetric manifold

$$\mathbf{O}_{I_4>0}/\mathbb{C} \cong \widehat{\mathbf{O}}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}\times U(1)}. \tag{42}$$

2. The unique duality orbit determined by the G-invariant constraint $I_4<0$ is the 55-

dimensional non-symmetric coset

$$\mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}, \tag{43}$$

with pseudo-Euclidean signature given by $(n_+, n_-) = (28, 27)$, thus with character $\chi = 0$. In this case, $M_-(\mathcal{Q})$ given by (39) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special pseudo-Kähler non-symmetric manifold

$$\mathbf{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+, \tag{44}$$

with signature $(n_+, n_-) = (28, 28)$. Through a "pseudo-conification" procedure (amounting to modding out $\mathbb{C}_s \cong SO(1,1) \times SO(1,1) \cong \mathbb{R}^+ \times \mathbb{R}^+$, one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special pseudo-Kähler symmetric manifold

$$\mathbf{O}_{I_4<0}/\mathbb{C}_s \cong \widehat{\mathbf{O}}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)} \times SO(1,1)}. \tag{45}$$

(41) and (44) are non-compact, real forms of $\frac{E_7}{E_6} \times GL(1)$, which is the type 29 in the classification of regular, pre-homogeneous vector spaces (PVS) worked out by Sato and Kimura in [34]. From its definition, a PVS is a finite-dimensional vector space V together with a subgroup G of $GL(V)$, such that G has an open dense orbit in V. PVS are subdivided into two types (type 1 and type 2), according to whether there exists an homogeneous polynomial on V which is invariant under the semi-simple (reductive) part of G itself. For more details, see *e.g.* [30, 35, 36].

In the case of $\frac{E_7}{E_6} \times GL(1)$, V is provided by the fundamental representation space $\mathbf{R} = \mathbf{56}$ of $G = E_7$, and there exists a quartic E_7-invariant polynomial I_4 (24) in the $\mathbf{56}$; $H = E_6$ is the isotropy (stabilizer) group.

Amazingly, simple, non-degenerate groups of type E_7 (relevant to $D = 4$ Einstein (super)gravities with symmetric scalar manifolds) *almost* saturate the list of irreducible PVS with unique G-invariant polynomial of degree 4 (*cfr.* Table 2); in particular, the parameter n characterizing each PVS can be interpreted as the number of centers of the regular solution in the (super)gravity theory with electric-magnetic duality (U-duality) group given by G. This topic will be considered in detail in a forthcoming publication.

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